3

PISA for Development Mathematics Framework

This chapter defines "mathematical literacy" as assessed in the Programme for International Student Assessment (PISA) and the extensions to the PISA mathematics framework that have been designed for the PISA for Development (PISA-D) project. It explains the processes, content knowledge and contexts reflected in PISA-D's mathematics problems, and provides several sample items. The chapter also discusses how student performance in mathematics is measured and reported.

WHAT IS NEW IN PISA-D? EXTENSIONS TO THE PISA MATHEMATICAL LITERACY FRAMEWORK

The objective of the PISA-D mathematics framework is to extend the PISA framework, in order to measure mathematical skills of students who perform at or below the lowest level on the standard math PISA scale. The outcomes of such measurement should provide reliable data that could help to plan the most effective ways of improving those students' mathematical skills. This extended framework is applicable, not only for students, but also for 14-16 year-olds who are out of school or not enrolled in PISA's target grades (Grade 7 or above).

To achieve this objective of the framework, it would seem natural to concentrate on some very basic "numeracy skills", such as fluency in performing simple arithmetical operations. However, it would not be an effective solution. While certainly some of these skills are needed to perform at the highest levels of PISA – such as arithmetical fluency, understanding of basic mathematical concepts, being able to recognise and identify graphs, and understanding of math vocabulary – they are not the focus of PISA.

The implementation of the PISA measurement of mathematics was preceded by a scientific discussion on the role of teaching mathematics. The references section of this chapter lists the most important scientific publications behind those discussions. The framework itself provides a comprehensive explanation of the most important conclusions, culminating in the concept of mathematical literacy. In short, it stresses the primary importance of the ability to use mathematics in a wide variety of contexts. The international success of PISA confirms this as a widely accepted way of understanding the primary goal of learning mathematics in today's world.

From this perspective, mastering the most basic technical skills is not enough. While it is important to be able to perform arithmetical operations, performing these operations it is not sufficient to get by mathematically in real life. To put this knowledge to use, one necessarily needs at least the basic skills of choosing the right model and selecting a strategy or an explanation. These skills constitute the core of the PISA understanding of mathematical literacy.

Identifying some of even the most basic technical skills as a measure of mathematical competencies would be, in this context, quite misleading. It could direct the attention of the users of the PISA results toward those skills as the primary education target, giving little chance to their students of becoming more mathematically literate.

The PISA-D mathematics framework adheres to the core idea of mathematical literacy, as defined by PISA. Therefore it is designed as an extension to the PISA 2015 mathematics framework, essentially measuring the same basic skills. The extensions aim to expand coverage at the lower ability levels in two ways: by using more straightforward, simply formulated items; and by suggesting a very careful analysis of students' attempts to solve the problem. The items used in PISA-D will also test the ability to choose the right model and select a strategy or an explanation. Thus PISA-D gains the potential to help improve students' mathematical literacy.

The extensions made to the PISA 2015 framework in PISA-D are an attempt to gain more information about students who currently perform below Level 1. In the mathematics framework, these extensions occur in three locations: descriptions of the proficiencies, where proficiency Level 1 was renamed as 1a and two new proficiency levels were added, 1b and 1c; adding five new activities to the process descriptors; and adding four new skills to the table relating the mathematical processes to the fundamental mathematical capabilities.

The PISA 2015 framework (OECD, 2016) continues the description and illustration of the PISA mathematics assessment as set out in the 2012 framework, when mathematics was re-examined and updated for use as the major domain in that cycle.

For PISA 2015, the computer was the primary mode of delivery for all domains, including mathematical literacy. The 2015 framework was updated to reflect the change in delivery mode, and includes a discussion of the considerations of transposing paper items to a screen and examples of what the results look like. The definition and constructs of mathematical literacy however, remain unchanged and consistent with those used in 2012. It is important to note that PISA-D includes a paper-based test for the in-school population and a tablet-based test for the out-of-school population. For this reason, therefore, the sections in this chapter dealing with computer-based assessment of mathematics only apply to the out-of-school assessment.

This chapter is organised into three major sections. The first section, "Defining mathematical literacy", explains the theoretical underpinnings of the PISA mathematics assessment, including the formal definition of the mathematical literacy construct. The second section, "Organising the domain of mathematics", describes three aspects: i) the mathematical *processes*

and the *fundamental mathematical capabilities* (in previous frameworks the "competencies") underlying those processes; ii) the way mathematical *content* knowledge is organised in the PISA 2015 framework, and the content knowledge that is relevant to an assessment of 15-year-old students; and iii) the *contexts* in which students will face mathematical challenges. The third section, "Assessing mathematical literacy", outlines the approach taken to apply the elements of the framework previously described, including the response formats, item scoring, reporting proficiency, testing mathematical literacy among the out-of-school population and examples of items for addressing the extended PISA-D framework.

DEFINING MATHEMATICAL LITERACY

An understanding of mathematics is central to a young person's preparedness for life in modern society. A growing proportion of problems and situations encountered in daily life, including in professional contexts, require some level of understanding of mathematics, mathematical reasoning and mathematical tools, before they can be fully understood and addressed. Mathematics is a critical tool for young people as they confront issues and challenges in personal, occupational, societal, and scientific aspects of their lives. It is thus important to have an understanding of the degree to which young people emerging from school are adequately prepared to apply mathematics to understanding important issues and solving meaningful problems. An assessment at age 15 provides an early indication of how individuals may respond in later life to the diverse array of situations they will encounter that involve mathematics.

The construct of mathematical literacy used in this chapter is based on PISA 2015, and is intended to describe the capacities of individuals to reason mathematically and use mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. This conception of mathematical literacy supports the importance of students developing a strong understanding of concepts of pure mathematics and the benefits of being engaged in explorations in the abstract world of mathematics. The construct of mathematical literacy, as defined for PISA, strongly emphasises the need to develop students' capacity to use mathematics in context, and it is important that they have rich experiences in their mathematics classrooms to accomplish this. For PISA 2012, mathematical literacy was defined as shown in Box 3.1. This is also the definition used in the PISA 2015 and PISA-D assessments.

Box 3.1 **The PISA 2015 definition of mathematical literacy**

Mathematical literacy is an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens.

The focus of the language in the definition of mathematical literacy is on active engagement in mathematics, and is intended to encompass reasoning mathematically and using mathematical concepts, procedures, facts and tools in describing, explaining and predicting phenomena. In particular, the verbs "formulate", "employ" and "interpret" point to the three processes in which students as active problem solvers will engage.

The language of the definition was also intended to integrate the notion of mathematical modelling, which has historically been a cornerstone of the PISA framework for mathematics (e.g. OECD, 2004), into the PISA 2015 definition of mathematical literacy. As individuals use mathematics and mathematical tools to solve problems in contexts, their work progresses through a series of stages (individually developed later in the document).

The modelling cycle is a central aspect of the PISA conception of students as active problem solvers; however, it is often not necessary to engage in every stage of the modelling cycle, especially in the context of an assessment (Niss et al., 2007). The problem solver frequently carries out some steps of the modelling cycle but not all of them, (e.g. when using graphs), or goes around the cycle several times to modify earlier decisions and assumptions.

The definition also acknowledges that mathematical literacy helps individuals to recognise the role that mathematics plays in the world and in helping they make the kinds of well-founded judgments and decisions required of constructive, engaged and reflective citizens.

Mathematical tools mentioned in the definition refer to a variety of physical and digital equipment, software and calculation devices. The 2015 computer-based survey, as well as the PISA-D tablet-based test, included an online calculator as part of the computer-based test material provided for some questions.

ORGANISING THE DOMAIN OF MATHEMATICS

The PISA mathematics framework defines the domain of mathematics for the PISA survey and describes an approach to the assessment of the mathematical literacy of 15-year-olds. That is, PISA assesses the extent to which 15-year-old students can handle mathematics adeptly when confronted with situations and problems – the majority of which are presented in real-world contexts.

For purposes of the assessment, the PISA 2015 definition of mathematical literacy can be analysed in terms of three inter-related aspects:

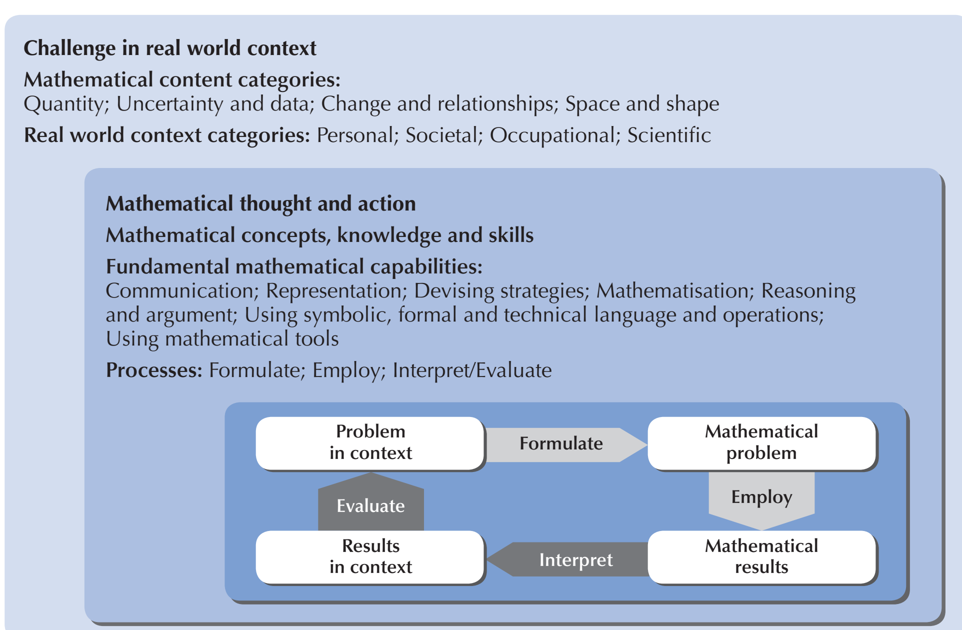
- the mathematical processes that describe what individuals do to connect the context of the problem with mathematics and thus solve the problem, and the capabilities that underlie those processes
- the mathematical content that is targeted for use in the assessment items
- the contexts in which the assessment items are located.

The following sections elaborate these aspects. In highlighting these aspects of the domain, the PISA 2012 mathematics framework, which was also used in PISA 2015 and PISA-D, helps ensure that assessment items developed for the survey reflect a range of processes, content, and contexts, so that, considered as a whole, the set of assessment items effectively operationalises what this framework defines as mathematical literacy. To illustrate the aspects of mathematical literacy, examples are available in the *PISA 2012 Assessment and Analytical Framework* (OECD, 2013) and on the PISA website (www.oecd.org/pisa).

Three questions, based on the PISA 2015 definition of mathematical literacy, lie behind the organisation of this section of the chapter. They are:

- What processes do individuals engage in when solving contextual mathematical problems, and what capabilities do we expect individuals to be able to demonstrate as their mathematical literacy grows?
- What mathematical content knowledge can we expect of individuals – and of 15-year-old students in particular?
- In what contexts can mathematical literacy be observed and assessed?

Figure 3.1 ▪ **A model of mathematical literacy in practice**



Source: OECD (2013), *PISA 2012 Assessment and Analytical Framework*, http://dx.doi.org/10.1787/9789264190511-en.

Mathematical processes and the underlying mathematical capabilities

Mathematical processes

The definition of mathematical literacy refers to an individual's capacity to formulate, employ and interpret mathematics. These three words, formulate, employ and interpret, provide a useful and meaningful structure for organising the mathematical processes that describe what individuals do to connect the context of a problem with the mathematics and thus solve the problem. Items in the PISA 2015 and the PISA-D mathematics survey will be assigned to one of three mathematical processes:

- formulating situations mathematically
- employing mathematical concepts, facts, procedures and reasoning
- interpreting, applying and evaluating mathematical outcomes.

It is important for both policy makers and those engaged more closely in the day-to-day education of students to know how effectively students are able to engage in each of these processes. The *formulating* process indicates how effectively students are able to recognise and identify opportunities to use mathematics in problem situations and then provide the necessary mathematical structure needed to formulate that contextualised problem into a mathematical form. The *employing* process indicates how well students are able to perform computations and manipulations and apply the concepts and facts that they know to arrive at a mathematical solution to a problem formulated mathematically. The *interpreting* process indicates how effectively students are able to reflect upon mathematical solutions or conclusions, interpret them in the context of a real-world problem, and determine whether the results or conclusions are reasonable. Students' facility at applying mathematics to problems and situations is dependent on skills inherent in all three of these processes, and an understanding of their effectiveness in each category can help inform both policy-level discussions and decisions being made closer to the classroom level.

In an effort to better measure the capabilities of Level 1b and 1c students, specific extensions have been made in PISA-D to the descriptions of the processes formulate, employ and interpret. The five additions are intended to better describe students' attempts to apply mathematical processes. This approach acknowledges that before students may be fully capable of utilising processes, they must first be able to identify and select an appropriate model, strategy or argument.

Formulating situations mathematically

The word *formulate* in the mathematical literacy definition refers to individuals being able to recognise and identify opportunities to use mathematics and then provide mathematical structure to a problem presented in some contextualised form. In the process of formulating situations mathematically, individuals determine where they can extract the essential mathematics to analyse, set up and solve the problem. They translate from a real-world setting to the domain of mathematics and provide the real-world problem with mathematical structure, representations and specificity. They reason about and make sense of constraints and assumptions in the problem. Specifically, this process of formulating situations mathematically includes activities such as the following:

- identifying the mathematical aspects of a problem situated in a real-world context and identifying the significant variables
- recognising mathematical structure (including regularities, relationships and patterns) in problems or situations
- simplifying a situation or problem in order to make it amenable to mathematical analysis
- identifying constraints and assumptions behind any mathematical modelling and simplifications gleaned from the context
- representing a situation mathematically, using appropriate variables, symbols, diagrams and standard models
- representing a problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions
- understanding and explaining the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically
- translating a problem into mathematical language or a representation
- recognising aspects of a problem that correspond with known problems or mathematical concepts, facts or procedures
- using technology (such as a spreadsheet or the list facility on a graphing calculator) to portray a mathematical relationship inherent in a contextualised problem.

In addition to the activities listed above, the following activity has been added to PISA-D:

- selecting an appropriate model from a list.

Employing mathematical concepts, facts, procedures and reasoning

The word *employ* in the mathematical literacy definition refers to individuals being able to apply mathematical concepts, facts, procedures and reasoning to solve mathematically formulated problems to obtain mathematical conclusions. In the process of employing mathematical concepts, facts, procedures and reasoning to solve problems, individuals perform the mathematical procedures needed to derive results and find a mathematical solution (e.g. performing arithmetic computations, solving equations, making logical deductions from mathematical assumptions, performing symbolic manipulations, extracting mathematical information from tables and graphs, representing and manipulating shapes in space, and analysing data). They work on a model of the problem situation, establish regularities, identify connections between mathematical entities and create mathematical arguments. Specifically, this process of employing mathematical concepts, facts, procedures and reasoning includes activities such as:

- devising and implementing strategies for finding mathematical solutions
- using mathematical tools, including technology, to help find exact or approximate solutions
- applying mathematical facts, rules, algorithms and structures when finding solutions
- manipulating numbers, graphical and statistical data and information, algebraic expressions and equations, and geometric representations
- making mathematical diagrams, graphs and constructions and extracting mathematical information from them
- using and switching between different representations in the process of finding solutions
- making generalisations based on the results of applying mathematical procedures to find solutions
- reflecting on mathematical arguments and explaining and justifying mathematical results.

In addition to the activities listed above, the following activities have been added to PISA-D:

- performing a simple calculation
- drawing a simple conclusion
- selecting an appropriate strategy from a list.

Interpreting, applying and evaluating mathematical outcomes

The word *interpret* used in the mathematical literacy definition focuses on the abilities of individuals to reflect upon mathematical solutions, results or conclusions and interpret them in the context of real-life problems. This involves translating mathematical solutions or reasoning back into the context of a problem and determining whether the results are reasonable and make sense in the context of the problem. This mathematical process category encompasses both the “interpret” and “evaluate” arrows noted in the previously defined model of mathematical literacy in practice (see Figure 3.1). Individuals engaged in this process may be called upon to construct and communicate explanations and arguments in the context of the problem, reflecting on both the modelling process and its results. Specifically, this process of interpreting, applying and evaluating mathematical outcomes includes activities such as:

- interpreting a mathematical result back into the real-world context
- evaluating the reasonableness of a mathematical solution in the context of a real-world problem
- understanding how the real-world impacts the outcomes and calculations of a mathematical procedure or model in order to make contextual judgments about how the results should be adjusted or applied
- explaining why a mathematical result or conclusion does, or does not, make sense given the context of a problem
- understanding the extent and limits of mathematical concepts and mathematical solutions
- critiquing and identifying the limits of the model used to solve a problem.

In addition to the activities listed above, the following activity has been added to PISA-D:

- evaluating a mathematical outcome in terms of the context.

Desired distribution of items by mathematical process

The goal in constructing the assessment is to achieve a balance that provides approximately equal weighting between the two processes that involve making a connection between the real world and the mathematical world and the process that calls for students to be able to work on a mathematically formulated problem. Table 3.1 shows the desired distribution of items by process for PISA 2015 and PISA-D (both for the in- and out-of-school instruments).

Table 3.1 Desired distribution of mathematics items, by process category

Process category	Percentage of items in PISA 2015	Percentage of items in PISA-D
Formulating situations mathematically	25	25
Employing mathematical concepts, facts, procedures and reasoning	50	50
Interpreting, applying and evaluating mathematical outcomes	25	25
Total	**100**	**100**

The desired distribution specifies the blueprint for selecting items according to important aspects of the domain frameworks. Item selection is based on the assessment design, as well as item characteristics related to a number of framework aspects – including process, content and context category, and consideration of the items' psychometric properties and appropriateness for this assessment. Following the assessment, the actual distributions of items across the framework aspects will be described in relation to the desired distributions. The extent to which the item pool for the assessment meets the framework specifications will be discussed in the technical report in the context of practical constraints in the item selection process.

Fundamental mathematical capabilities underlying the mathematical processes

A decade of experience in developing PISA items and analysing the ways in which students respond to items has revealed that there is a set of fundamental mathematical capabilities that underpins each of these reported processes and mathematical literacy in practice. The work of Mogens Niss and his Danish colleagues (Niss, 2003; Niss and Jensen, 2002; Niss and Højgaard, 2011) identified eight capabilities – referred to as "competencies" by Niss and in the 2003 framework (OECD, 2004) – that are instrumental to mathematical behaviour.

The PISA 2015 and PISA-D frameworks uses a modified formulation of this set of capabilities, which condenses the number from eight to seven based on an investigation of the operation of the competencies through previously administered PISA items (Turner et al., 2013). There is wide recognition of the need to identify such a set of general mathematical capabilities to complement the role of specific mathematical content knowledge in mathematics learning. Prominent examples include the eight mathematical practices of the Common Core State Standards Initiative in the United States (CCSSI, 2010), the four key processes (representing, analysing, interpreting and evaluating, and communicating and reflecting) of England's Mathematics National Curriculum (QCA, 2007), and the process standards in the United States' National Council of Teachers of Mathematics' Principles and Standards for School Mathematics (NCTM, 2000). These cognitive capabilities are available to or learnable by individuals in order to understand and engage with the world in a mathematical way, or to solve problems. As the level of mathematical literacy possessed by an individual increases, that individual is able to draw to an increasing degree on the fundamental mathematical capabilities (Turner and Adams, 2012). Thus, increasing activation of fundamental mathematical capabilities is associated with increasing item difficulty. This observation has been used as the basis of the descriptions of different proficiency levels of mathematical literacy reported in previous PISA surveys and discussed later in this framework.

The seven fundamental mathematical capabilities used in the PISA 2015 and PISA-D frameworks are as follows:

- *Communication*: Mathematical literacy involves communication. The individual perceives the existence of some challenge and is stimulated to recognise and understand a problem situation. Reading, decoding and interpreting statements, questions, tasks or objects enables the individual to form a mental model of the situation, which is an important step in understanding, clarifying and formulating a problem. During the solution process, intermediate results may need to be summarised and presented. Later on, once a solution has been found, the problem solver may need to present the solution, and perhaps an explanation or justification, to others.

- *Mathematising*: Mathematical literacy can involve transforming a problem defined in the real world to a strictly mathematical form (which can include structuring, conceptualising, making assumptions and/or formulating a model), or interpreting or evaluating a mathematical outcome or a mathematical model in relation to the original problem. The term mathematising is used to describe the fundamental mathematical activities involved.
- *Representation*: Mathematical literacy very frequently involves representations of mathematical objects and situations. This can entail selecting, interpreting, translating between, and using a variety of representations to capture a situation, interact with a problem, or to present one's work. The representations referred to include graphs, tables, diagrams, pictures, equations, formulae and concrete materials.
- *Reasoning and argument:* A mathematical ability that is called on throughout the different stages and activities associated with mathematical literacy is referred to as reasoning and argument. This capability involves logically rooted thought processes that explore and link problem elements so as to make inferences from them, check a justification that is given, or provide a justification of statements or solutions to problems.
- *Devising strategies for solving problems*: Mathematical literacy frequently requires devising strategies for solving problems mathematically. This involves a set of critical control processes that guide an individual to effectively recognise, formulate, and solve problems. This skill is characterised as selecting or devising a plan or strategy to use mathematics to solve problems arising from a task or context, as well as guiding its implementation. This mathematical capability can be demanded at any of the stages of the problem solving process.
- *Using symbolic, formal and technical language and operations*: Mathematical literacy requires using symbolic, formal, and technical language and operations. This involves understanding, interpreting, manipulating and making use of symbolic expressions within a mathematical context (including arithmetic expressions and operations) governed by mathematical conventions and rules. It also involves understanding and utilising formal constructs based on definitions, rules and formal systems and also using algorithms with these entities. The symbols, rules and systems used will vary according to what particular mathematical content knowledge is needed for a specific task to formulate, solve or interpret the mathematics.
- *Using mathematical tools:*[1] The final mathematical capability that underpins mathematical literacy in practice is using mathematical tools. Mathematical tools encompass physical tools such as measuring instruments, as well as calculators and computer-based tools that are becoming more widely available. This ability involves knowing about and being able to make use of various tools that may assist mathematical activity, and knowing about the limitations of such tools. Mathematical tools can also have an important role in communicating results.

These capabilities are evident to varying degrees in each of the three mathematical processes. The ways in which these capabilities manifest themselves within the three processes are described in Figure 3.2.

A good guide to the empirical difficulty of items can be obtained by considering which aspects of the fundamental mathematical capabilities are required for planning and executing a solution (Turner, 2012; Turner and Adams, 2012; Turner et al., 2013). The easiest items will require the activation of few capabilities and in a relatively straightforward way. The hardest items require complex activation of several capabilities. Predicting difficulty requires consideration of both the number of capabilities and the complexity of activation required. Based on the modifications to the proficiencies and processes for PISA-D, it was necessary to add particular skills to support these modifications. Four skills were added to the table in order to provide better understanding of the extensions of the mathematical process descriptions. These skills also support the capabilities delineated in the proficiencies 1b and 1c.

Figure 3.2 ▪ **Relationship between mathematical processes (top row) and fundamental mathematical capabilities (left-most column)**

	Formulating situations mathematically	Employing mathematical concepts, facts, procedures and reasoning	Interpreting, applying and evaluating mathematical outcomes
Communicating	Read, decode and make sense of statements, questions, tasks, objects or images, in order to form a mental model of the situation	Articulate a solution, show the work involved in reaching a solution, and/or summarise and present intermediate mathematical results	Construct and communicate explanations and arguments in the context of the problem
Mathematising	Identify the underlying mathematical variables and structures in the real-world problem, and make assumptions so that they can be used For PISA-D, "Select a model appropriate to the context of real-world problems" has been added	Use an understanding of the context to guide or expedite the mathematical solving process, e.g. working to a context-appropriate level of accuracy	Understand the extent and limits of a mathematical solution that are a consequence of the mathematical model employed
Representation	Create a mathematical representation of real-world information For PISA-D, "Select a representation appropriate to the context" has been added	Make sense of, relate and use a variety of representations when interacting with a problem	Interpret mathematical outcomes in a variety of formats in relation to a situation or use; compare or evaluate two or more representations in relation to a situation
Reasoning and argument	Explain, defend or provide a justification for the identified or devised representation of a real-world situation	Explain, defend or provide a justification for the processes and procedures used to determine a mathematical result or solution Connect pieces of information to arrive at a mathematical solution, make generalisations or create a multi-step argument For PISA-D, "Select an appropriate justification" has been added	Reflect on mathematical solutions and create explanations and arguments that support, refute or qualify a mathematical solution to a contextualised problem
Devising strategies for solving problems	Select or devise a plan or strategy to mathematically reframe contextualised problems	Activate effective and sustained control mechanisms across a multi-step procedure leading to a mathematical solution, conclusion or generalisation	Devise and implement a strategy in order to interpret, evaluate and validate a mathematical solution to a contextualised problem For PISA-D, "Implement a given strategy" has been added
Using symbolic, formal, and technical language and operations	Use appropriate variables, symbols, diagrams and standard models in order to represent a real-world problem using symbolic/formal language	Understand and utilise formal constructs based on definitions, rules and formal systems as well as employing algorithms	Understand the relationship between the context of the problem and representation of the mathematical solution Use this understanding to help interpret the solution in context and gauge the feasibility and possible limitations of the solution
Using mathematical tools	Use mathematical tools in order to recognise mathematical structures or to portray mathematical relationships	Know about and be able to make appropriate use of various tools that may assist in implementing processes and procedures for determining mathematical solutions	Use mathematical tools to ascertain the reasonableness of a mathematical solution and any limits and constraints on that solution, given the context of the problem

Mathematical content knowledge

An understanding of mathematical content – and the ability to apply that knowledge to the solution of meaningful contextualised problems – is important for citizens in the modern world. That is, to solve problems and interpret situations in personal, occupational, societal and scientific contexts, there is a need to draw upon certain mathematical knowledge and understandings.

Mathematical structures have been developed over time as a means to understand and interpret natural and social phenomena. In schools, the mathematics curriculum is typically organised around content strands (e.g. number, algebra and geometry) and detailed topic lists that reflect historically well-established branches of mathematics and that help in defining a structured curriculum. However, outside the mathematics classroom, a challenge or situation that arises is usually not accompanied by a set of rules and prescriptions that shows how the challenge can be met. Rather it typically requires some creative thought in seeing the possibilities of bringing mathematics to bear on the situation and in formulating it mathematically. Often a situation can be addressed in different ways, drawing on different mathematical concepts, procedures, facts or tools.

Since the goal of PISA is to assess mathematical literacy, an organisational structure for mathematical content knowledge has been developed based on the mathematical phenomena that underlie broad classes of problems and which have motivated the development of specific mathematical concepts and procedures. Because national mathematics curricula are typically designed to equip students with knowledge and skills that address these same underlying mathematical phenomena, the outcome is that the range of content arising from organising content this way is closely aligned with that typically found in national mathematics curricula. This framework lists some content topics appropriate for assessing the mathematical literacy of 15-year-old students based on analyses of national standards from eleven countries.

To organise the domain of mathematics for purposes of assessing mathematical literacy, it is important to select a structure that grows out of historical developments in mathematics, that encompasses sufficient variety and depth to reveal the essentials of mathematics, and that also represents, or includes, the conventional mathematical strands in an acceptable way. Thus, a set of content categories that reflects the range of underlying mathematical phenomena was selected for the PISA 2015 framework, consistent with the categories used for previous PISA surveys.

The following list of content categories, therefore, are used in PISA 2015 and PISA-D to meet the requirements of historical development, coverage of the domain of mathematics and the underlying phenomena that motivate its development, and reflection of the major strands of school curricula. These four categories characterise the range of mathematical content that is central to the discipline and illustrate the broad areas of content used in the test items for PISA 2015 and PISA-D:

- Change and relationships
- Space and shape
- Quantity
- Uncertainty and data

With these four categories, the mathematical domain can be organised in a way that ensures a spread of items across the domain and focuses on important mathematical phenomena, but at the same time, avoids a too fine division that would work against a focus on rich and challenging mathematical problems based on real situations. While categorisation by content category is important for item development and selection, and for reporting of assessment results, it is important to note that some specific content topics may materialise in more than one content category. For example, a released PISA item called *Pizzas* involves determining which of two round pizzas, with different diameters and different costs but the same thickness, is the better value. This item draws on several areas of mathematics, including measurement, quantification (value for money, proportional reasoning and arithmetic calculations), and change and relationships (in terms of relationships among the variables and how relevant properties change from the smaller pizza to the larger one.) This item was ultimately categorised as a *change and relationships* item since the key to the problem lies in students being able to relate the change in areas of the two pizzas (given a change in diameter) and a corresponding change of price. Clearly, a different item involving circle area might be classified as a *space and shape* item. Connections between aspects of content that span these four content categories contribute to the coherence of mathematics as a discipline and are apparent in some of the assessment items selected for the PISA 2015 assessment.

The broad mathematical content categories and the more specific content topics appropriate for 15-year-old students described later in this section reflect the level and breadth of content that is eligible for inclusion on the PISA 2015 and PISA-D surveys. Narrative descriptions of each content category and the relevance of each to solving meaningful

problems are provided first, followed by more specific definitions of the kinds of content that are appropriate for inclusion in an assessment of mathematical literacy of 15-year-old students and out-of-school youth. These specific topics reflect commonalities found in the expectations set by a range of countries and educational jurisdictions. The standards examined to identify these content topics are viewed as evidence not only of what is taught in mathematics classrooms in these countries but also as indicators of what countries view as important knowledge and skills for preparing students of this age to become constructive, engaged and reflective citizens.

Descriptions of the mathematical content knowledge that characterise each of the four categories — *change and relationship*s, *space and shape*, *quantity*, and *uncertainty and data* — are provided below.

Change and relationships

The natural and designed worlds display a multitude of temporary and permanent relationships among objects and circumstances, where changes occur within systems of inter-related objects or in circumstances where the elements influence one another. In many cases these changes occur over time, and in other cases changes in one object or quantity are related to changes in another. Some of these situations involve discrete change; others change continuously. Some relationships are of a permanent, or invariant, nature. Being more literate about change and relationships involves understanding fundamental types of change and recognising when they occur in order to use suitable mathematical models to describe and predict change. Mathematically this means modelling the change and the relationships with appropriate functions and equations, as well as creating, interpreting, and translating among symbolic and graphical representations of relationships.

Change and relationships is evident in such diverse settings as growth of organisms, music, and the cycle of seasons, weather patterns, employment levels, and economic conditions. Aspects of the traditional mathematical content of functions and algebra, including algebraic expressions, equations and inequalities, and tabular and graphical representations, are central in describing, modelling, and interpreting change phenomena. For example, the released PISA unit *Walking* (see the "Examples of items" section) contains two items that exemplify the *change and relationships* category since the focus is on the algebraic relationships between two variables, requiring students to activate their algebraic knowledge and skills. Students are required to employ a given formula for pacelength – a formula expressed in algebraic form – to determine pacelength in one item and walking speed in the other. Representations of data and relationships described using statistics also are often used to portray and interpret change and relationships, and a firm grounding in the basics of number and units is also essential to defining and interpreting *change and relationships*. Some interesting relationships arise from geometric measurement, such as the way that changes in perimeter of a family of shapes might relate to changes in area, or the relationships among lengths of the sides of triangles.

Space and shape

Space and shape encompasses a wide range of phenomena that are encountered everywhere in our visual and physical world: patterns, properties of objects, positions and orientations, representations of objects, decoding and encoding of visual information, and navigation and dynamic interaction with real shapes as well as with representations. Geometry serves as an essential foundation for *space and shape*, but the category extends beyond traditional geometry in content, meaning, and method, drawing on elements of other mathematical areas such as spatial visualisation, measurement and algebra. For instance, shapes can change, and a point can move along a locus, thus requiring function concepts. Measurement formulae are central in this area. The manipulation and interpretation of shapes in settings that call for tools ranging from dynamic geometry software to global positioning system (GPS) software are included in this content category.

PISA assumes that the understanding of a set of core concepts and skills is important to mathematical literacy relative to *space and shape*. Mathematical literacy in the area of *space and shape* involves a range of activities such as understanding perspective (for example in paintings), creating and reading maps, transforming shapes with and without technology, interpreting views of three-dimensional scenes from various perspectives and constructing representations of shapes.

Quantity

The notion of *quantity* may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, our world. It incorporates the quantification of attributes of objects, relationships, situations and entities in the world, understanding various representations of those quantifications, and judging interpretations and arguments based on quantity. To engage with the quantification of the world involves understanding measurements, counts, magnitudes, units, indicators, relative size, and numerical trends and patterns. Aspects of quantitative reasoning—such as number

sense, multiple representations of numbers, elegance in computation, mental calculation, estimation and assessment of reasonableness of results—are the essence of mathematical literacy relative to *quantity*.

Quantification is a primary method for describing and measuring a vast set of attributes of aspects of the world. It allows for the modelling of situations, for the examination of change and relationships, for the description and manipulation of space and shape, for organising and interpreting data, and for the measurement and assessment of uncertainty. Thus mathematical literacy in the area of *quantity* applies knowledge of number and number operations in a wide variety of settings.

Uncertainty and data

In science, technology and everyday life, uncertainty is a given. Uncertainty is therefore a phenomenon at the heart of the mathematical analysis of many problem situations, and the theory of probability and statistics as well as techniques of data representation and description have been established to deal with it. The *uncertainty and data* content category includes recognising the place of variation in processes, having a sense of the quantification of that variation, acknowledging uncertainty and error in measurement, and knowing about chance. It also includes forming, interpreting and evaluating conclusions drawn in situations where uncertainty is central. The presentation and interpretation of data are key concepts in this category (Moore, 1997).

There is uncertainty in scientific predictions, poll results, weather forecasts and economic models. There is variation in manufacturing processes, test scores and survey findings, and chance is fundamental to many recreational activities enjoyed by individuals. The traditional curricular areas of probability and statistics provide formal means of describing, modelling, and interpreting a certain class of uncertainty phenomena, and for making inferences. In addition, knowledge of number and of aspects of algebra such as graphs and symbolic representation contribute to facility in engaging in problems in this content category. The focus on the interpretation and presentation of data is an important aspect of the *uncertainty and data* category.

Desired distribution of items by content category

The desired distribution of items selected for PISA 2015 and for PISA-D (both in- and out-of-school instruments) across the four content categories is shown in Table 3.2. The goal in constructing the survey is a balanced distribution of items with respect to content category, since all of these domains are important for constructive, engaged and reflective citizens.

Table 3.2 Desired distribution of mathematics items, by content category

Content category	Percentage of items in PISA 2015	Percentage of items in PISA-D
Change and relationships	25	25
Space and shape	25	25
Quantity	25	25
Uncertainty and data	25	25
Total	**100**	**100**

Content topics for guiding the assessment of mathematical literacy for 15-year-old students

To effectively understand and solve contextualised problems involving *change and relationships, space and shape, quantity,* and *uncertainty and data* requires drawing upon a variety of mathematical concepts, procedures, facts and tools at an appropriate level of depth and sophistication. As an assessment of mathematical literacy, PISA strives to assess the levels and types of mathematics that are appropriate for 15-year-old students on a trajectory to become constructive, engaged and reflective citizens able to make well-founded judgments and decisions. It is also the case that PISA, while not designed or intended to be a curriculum-driven assessment, strives to reflect the mathematics that students have likely had the opportunity to learn by the time they are 15 years old.

The content included in PISA-D and PISA 2015 is the same as that developed in PISA 2012.The four content categories of *change and relationships, space and shape, quantity,* and *uncertainty and data* serve as the foundation for identifying this range of content, yet there is not a one-to-one mapping of content topics to these categories. For example, proportional reasoning comes into play in such varied contexts as making measurement conversions, analysing linear relationships, calculating probabilities and examining the lengths of sides in similar shapes. The following content is intended to reflect

the centrality of many of these concepts to all four content categories and reinforce the coherence of mathematics as a discipline. It intends to be illustrative of the content topics included in PISA-D, rather than an exhaustive listing:

- *Functions*: the concept of function, emphasising but not limited to linear functions, their properties, and a variety of descriptions and representations of them. Commonly used representations are verbal, symbolic, tabular and graphical.
- *Algebraic expressions:* verbal interpretation of and manipulation with algebraic expressions, involving numbers, symbols, arithmetic operations, powers and simple roots.
- *Equations and inequalities:* linear and related equations and inequalities, simple second-degree equations, and analytic and non-analytic solution methods.
- *Co-ordinate systems:* representation and description of data, position and relationships.
- *Relationships within and among geometrical objects in two and three dimensions:* static relationships such as algebraic connections among elements of figures (e.g. the Pythagorean theorem as defining the relationship between the lengths of the sides of a right triangle), relative position, similarity and congruence, and dynamic relationships involving transformation and motion of objects, as well as correspondences between two- and three-dimensional objects.
- *Measurement:* quantification of features of and among shapes and objects, such as angle measures, distance, length, perimeter, circumference, area, and volume.
- *Numbers and units:* concepts; representations of numbers and number systems, including properties of integer and rational numbers; relevant aspects of irrational numbers; as well as quantities and units referring to phenomena such as time, money, weight, temperature, distance, area and volume, and derived quantities and their numerical description.
- *Arithmetic operations:* the nature and properties of these operations and related notational conventions.
- *Percents, ratios and proportions:* numerical description of relative magnitude and the application of proportions and proportional reasoning to solve problems.
- *Counting principle*s: simple combinations and permutations.
- *Estimation:* purpose-driven approximation of quantities and numerical expressions, including significant digits and rounding.
- *Data collection, representation and interpretation:* nature, genesis, and collection of various types of data, and the different ways to represent and interpret them.
- *Data variability and its description:* concepts such as variability, distribution, and central tendency of data sets, and ways to describe and interpret these in quantitative terms.
- *Samples and sampling:* concepts of sampling and sampling from data populations, including simple inferences based on properties of samples.
- *Chance and probability:* notion of random events, random variation and its representation, chance and frequency of events, and basic aspects of the concept of probability.

Contexts

The choice of appropriate mathematical strategies and representations is often dependent on the context in which a problem arises. Being able to work within a context is widely appreciated to place additional demands on the problem solver (see Watson and Callingham, 2003, for findings about statistics). For the PISA survey, it is important that a wide variety of contexts are used. This offers the possibility of connecting with the broadest possible range of individual interests and with the range of situations in which individuals operate in the 21st century.

For purposes of the PISA-D mathematics framework, four context categories have been defined and are used to classify assessment items developed for the PISA survey:

- *Personal* – Problems classified in the personal context category focus on activities of one's self, one's family or one's peer group. The kinds of contexts that may be considered personal include (but are not limited to) those involving food preparation, shopping, games, personal health, personal transportation, sports, travel, personal scheduling and personal finance.
- *Occupational* – Problems classified in the occupational context category are centred on the world of work. items categorised as occupational may involve (but are not limited to) such things as measuring, costing and ordering materials for building, payroll/accounting, quality control, scheduling/inventory, design/architecture and job-related decision making. Occupational contexts may relate to any level of the workforce, from unskilled work to the highest levels of professional work, although items in the PISA survey must be accessible to 15-year-old students.

- *Societal* – Problems classified in the societal context category focus on one's community (whether local, national or global). They may involve (but are not limited to) such things as voting systems, public transport, government, public policies, demographics, advertising, national statistics and economics. Although individuals are involved in all of these things in a personal way, in the societal context category, the focus of problems is on the community perspective.
- *Scientific* – Problems classified in the scientific category relate to the application of mathematics to the natural world and issues and topics related to science and technology. Particular contexts might include (but are not limited to) such areas as weather or climate, ecology, medicine, space science, genetics, measurement and the world of mathematics itself. Items that are intramathematical, where all the elements involved belong in the world of mathematics, fall within the scientific context.

PISA items were arranged in units that share stimulus material. It is therefore usually the case that all items in the same unit belonged to the same context category. Exceptions do arise; for example stimulus material may be examined from a personal point of view in one item and a societal point of view in another. When an item involved only mathematical constructs without reference to the contextual elements of the unit within which it is located, it was allocated to the context category of the unit. In the unusual case of a unit involving only mathematical constructs and being without reference to any context outside of mathematics, the unit was assigned to the scientific context category.

Using these context categories provided the basis for selecting a mix of item contexts and ensures that the assessment reflects a broad range of uses of mathematics, ranging from everyday personal uses to the scientific demands of global problems. Moreover it was important that each context category be populated with assessment items having a broad range of item difficulties. Given that the major purpose of these context categories is to challenge students in a broad range of problem contexts, each category was designed to contribute substantially to the measurement of mathematical literacy. It should not be the case that the difficulty level of assessment items representing one context category is systematically higher or lower than the difficulty level of assessment items in another category.

In identifying contexts that may be relevant, it is critical to keep in mind that a purpose of the assessment is to gauge the use of mathematical content knowledge, processes and capabilities that students have acquired by age 15. Contexts for assessment items, therefore, were selected in light of relevance to students' interests and lives, and the demands that will be placed upon them as they enter society as constructive, engaged and reflective citizens. National Project Managers from countries participating in the PISA-D survey were involved in judging the degree of such relevance.

Desired distribution of items by context category

The desired distribution of items selected for PISA 2015 and for PISA-D (both in- and out-of-school instruments) across the four content categories is shown in Table 3.3. With this balanced distribution, no single context type is allowed to dominate, providing students with items that span a broad range of individual interests and a range of situations that they might expect to encounter in their lives.

Table 3.3 Desired distribution of mathematics items, by context category

Context category	Percentage of items in PISA 2015	Percentage of items in PISA-D
Personal	25	25
Occupational	25	25
Societal	25	25
Scientific	25	25
Total	**100**	**100**

ASSESSING MATHEMATICAL LITERACY

This section outlines the approach taken to apply the elements of the framework described in previous sections to PISA 2015 and PISA-D. This includes the structure of the mathematics component of the PISA-D survey, arrangements for transferring the paper-based trend items to a computer-based delivery, and reporting mathematical proficiency.

Response formats

Three types of response format are used to assess mathematical literacy in PISA 2015 and PISA-D: open constructed-response, closed constructed-response and selected-response (simple and complex multiple-choice) items. Open constructed-response items require a somewhat extended written response from a student. Such items also may ask

the student to show the steps taken or to explain how the answer was reached. These items require trained experts to manually code student responses.

Closed constructed-response items provide a more structured setting for presenting problem solutions, and they produce a student response that can be easily judged to be either correct or incorrect. Often student responses to questions of this type can be keyed into data capture software, and coded automatically, but some must be manually coded by trained experts. The most frequent closed constructed-responses are single numbers.

Selected-response items require students to choose one or more responses from a number of response options. Responses to these questions can usually be automatically processed. About equal numbers of each of these response formats is used to construct the survey instruments.

Item scoring

Although the majority of the items were dichotomously scored (that is, responses are awarded either credit or no credit), the open constructed-response items can sometimes involve partial credit scoring, which allows responses to be assigned credit according to differing degrees of “correctness” of responses. For each such item, a detailed coding guide that allows for full credit, partial credit, or no credit was provided to persons trained in the coding of student responses across the range of participating countries to ensure coding of responses was done in a consistent and reliable way. To maximise the comparability between the paper-based and computer-based assessment, careful attention is given to the scoring guides in order to ensure that the important elements were included.

Reporting proficiency in mathematics

The outcomes of the PISA mathematics survey are reported in a number of ways. Estimates of overall mathematical proficiency are obtained for sampled students in each participating country, and a number of proficiency levels are defined. Descriptions of the degree of mathematical literacy typical of students in each level are also developed. For PISA 2003, scales based on the four broad content categories were developed. In Figure 3.3, descriptions for the six proficiency levels reported for the overall PISA mathematics scale in 2012 are presented. These form the basis for the PISA 2015 mathematics scale and the PISA-D mathematics scale. The finalised 2012 scale is used to report the PISA 2015 outcomes. For PISA-D, in addition, the existing Level 1 was renamed Level 1a, and the table describing the proficiencies has been extended to include Levels 1b and 1c.

Fundamental mathematical capabilities play a central role in defining what it means to be at different levels of the scales for mathematical literacy overall and for each of the reported processes. For example, in the proficiency scale description for Level 4 (see Figure 3.3), the second sentence highlights aspects of mathematising and representation that are evident at this level. The final sentence highlights the characteristic communication, reasoning and argument of Level 4, providing a contrast with the short communications and lack of argument of Level 3 and the additional reflection of Level 5. In an earlier section of this framework and in Figure 3.2, each of the mathematical processes was described in terms of the fundamental mathematical capabilities that individuals might activate when engaging in that process.

Figure 3.3 ▪ **Summary description of the eight levels of mathematics proficiency in PISA-D**

Level	Lower score limit	Percentage of students across OECD countries at each level, PISA 2015	Percentage of students across 23 middle- and low-income countries at each level, PISA 2015	Descriptor
6	669	2.3%	0.3%	At Level 6, students can conceptualise, generalise and utilise information based on their investigations and modelling of complex problem situations, and can use their knowledge in relatively non-standard contexts. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply this insight and understanding, along with a mastery of symbolic and formal mathematical operations and relationships, to develop new approaches and strategies for attacking novel situations. Students at this level can reflect on their actions, and can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments and the appropriateness of these to the original situation.

Figure 3.3 [continued] ■ **Summary description of the eight levels of mathematics proficiency in PISA-D**

Level	Lower score limit	Percentage of students across OECD countries at each level, PISA 2015	Percentage of students across 23 middle- and low-income countries at each level, PISA 2015	Descriptor
5	607	8.4%	1.5%	At Level 5, students can develop and work with models for complex situations, identifying constraints and specifying assumptions. They can select, compare and evaluate appropriate problem-solving strategies for dealing with complex problems related to these models. Students at this level can work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterisations, and insight pertaining to these situations. They begin to reflect on their work and can formulate and communicate their interpretations and reasoning.
4	545	18.6%	5.3%	At Level 4, students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions. They can select and integrate different representations, including symbolic, linking them directly to aspects of real-world situations. Students at this level can utilise their limited range of skills and can reason with some insight, in straightforward contexts. They can construct and communicate explanations and arguments based on their interpretations, arguments and actions.
3	482	24.8%	12.6%	At Level 3, students can execute clearly described procedures, including those that require sequential decisions. Their interpretations are sufficiently sound to be a base for building a simple model or for selecting and applying simple problem-solving strategies. Students at this level can interpret and use representations based on different information sources and reason directly from them. They typically show some ability to handle percentages, fractions and decimal numbers, and to work with proportional relationships. Their solutions reflect that they have engaged in basic interpretation and reasoning.
2	420	22.5%	21.6%	At Level 2, students can interpret and recognise situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formulae, procedures or conventions to solve problems involving whole numbers. They are capable of making literal interpretations of the results.
1a	358	14.9%	26.3%	At Level 1a, students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are almost always obvious and follow immediately from the given stimuli.
1b	295	8.5% (percentage of students scoring below Level 1, PISA 2015)	32.4% (percentage of students scoring below Level 1, PISA 2015)	At Level 1b, students can respond to questions involving easy to understand contexts where all relevant information is clearly given in a simple representation (for example tabular or graphic) and defined in a short syntactically simple text. They are able to follow clearly prescribed instructions.
1c	236			At Level 1c, students can respond to questions involving easy to understand contexts where all relevant information is clearly given in a simple, familiar format (for example a small table or picture) and defined in a very short syntactically simple text. They are able to follow a clear instruction describing a single step or operation.

Note: Descriptors 2 through 6 are the same as those used in PISA 2012, and Level 1 was renamed Level 1a.

In order to gain useful information for these new levels, 1b and 1c, it is vital that context and language do not interfere with the mathematics being assessed. To this end, the context and language must be carefully considered.

The context for both 1b and 1c should be situations that students encounter on a daily basis. Examples of these contexts may include money, temperature, food, time, date, weight, size and distance. All items should be concrete and not abstract. The focus of the item should be mathematical only. The understanding of the context should not interfere with the performance of the item.

Equally important is to have all items formulated in the simplest possible terms. Sentences should be short and direct. Compound sentences, compound nouns and conditional sentences should be avoided. Vocabulary used in the items must be carefully examined to ensure that students will have a clear understanding of what is being required. In addition, special care will be given to ensure that no extra difficulty is added due to a heavy text load or by a context that is unfamiliar to students based on their cultural background.

Items designed for Level 1c should only ask for a single step or operation. However, it is important to note that a single step or operation is not limited to an arithmetical step. This step might be demonstrated by making a selection or identifying some information. All parts of the modelling cycle can and should be used to measure the mathematical abilities of students at Levels 1b and 1c.

Testing mathematical literacy among the out-of-school population

For the out-of-school population, item selection focused on the scale at or below Level 2 with an emphasis on the lower end of the scale in terms of item distribution. The selection process was similar to that used for the in-school population: coverage of all processes was maintained and contexts of the items were reviewed to ensure appropriateness for what individuals would encounter in an out-of-school context.

Box 3.2 **Delivery mode**

The PISA-D school-based assessment is paper-based, while the out-of-school assessment is conducted on a tablet computer. To ensure comparability between the tests, the tablet-based instruments for PISA-D are formed by a subgroup of the items used for the paper-based assessment. All these items were originally designed for a paper-based assessment, so when moving to a tablet-based delivery, care was taken to maintain comparability between the assessments. The PISA 2015 framework describes some factors that must be considered when transposing items from paper to computer mode. These elements were also taken into account when designing the out-of-school instrument for PISA-D.

Item types: The computer provides a range of opportunities for designers of test items, including new item formats (e.g. drag-and-drop, hotspots). Since the PISA-D tablet-based tests use a subgroup of items from the paper-based test, there is less opportunity to exploit innovative item types and the majority of response formats remains unchanged.

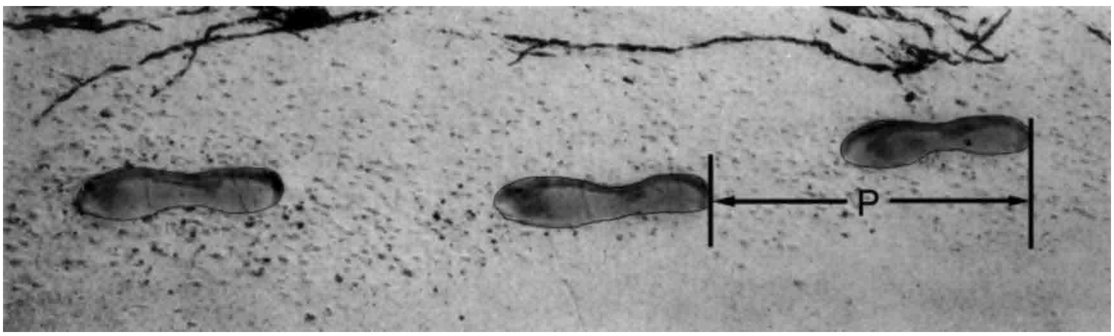
Stimulus presentation: A feature of fixed texts defined in the construct is that “the extent or amount of the text is immediately visible to the reader”. Clearly, it is impossible, both on paper and on a screen, to have long texts displayed on a single page or screen. To allow for this and still satisfy the construct of fixed texts, pagination is used for texts rather than scrolling. Texts that cover more than one page are presented in their entirety before the student sees the first question.

IT skills: Just as paper-based assessments rely on a set of fundamental skills for working with printed materials, so computer-based assessments rely on a set of fundamental information and communications technology skills for using computers. These include knowledge of basic hardware (e.g. keyboard and mouse) and basic conventions (e.g. arrows to move forward and specific buttons to press to execute commands). The intention is to keep such skills to a minimal core level in the tablet-based assessment.

Examples of items for addressing the extended PISA-D mathematics framework

The following items illustrate proficiency Levels 1a, 1b and 1c. In each case the tasks involved are described and an explanation is given about why the item fits a certain proficiency level. The items either come from or are adapted from PISA or work carried out by the Institute of Educational Research in Poland.

Sample item 1



The picture shows the footprints of a man walking. The pacelength P is the distance between the rear of two consecutive footprints.

For men, the formula $\frac{n}{P} = 140$ gives an approximate relationship between n and P where

n = number of steps per minute, and
P = pacelength in metres.

Heiko has a pacelength that is 0.5 metres. Using this formula, how many steps per minute, n, does Heiko take each minute?

For this item, the student simply needs to substitute the values into the formula and solve it. Since the pacelength is given (0.5 metres), the student only needs to complete a single operational step. Multiplying both sides of the equation by 0.5 gives a value for n of 70. This addresses the added process "performing a simple calculation." A student who only substitutes the values correctly would meet the requirements of proficiency 1a. The abstract format of a formula involving two variables does not meet the requirements of 1b or 1c.

Sample item 2

Mei-Ling found out that the exchange rate between Singapore dollars and South African rand was

$$1 \text{ SGD} = 4.2 \text{ ZAR}$$

Mei-Ling changed 3 000 Singapore dollars into South African rand at this exchange rate. Choose a correct method from those listed. Then calculate n, the amount of South African rand Mei-Ling received after the exchange.

$$\frac{1}{4.2} = \frac{n}{3000} \qquad \frac{1}{3000} = \frac{4.2}{n} \qquad 4.2n = 3000 \qquad n = 3000(4.2)$$

For this item, the student is given four methods to solve for n. Two of these methods will result in a correct value for n. The expectation is that a student will be able to select one of the correct methods and then solve for the value of n. This addresses the added process, "selecting an appropriate model from a list." If a student is able to choose one of the correct methods, the requirements for proficiency 1b are met. If a student is also able to solve for n correctly, the requirements for proficiency 1a are met.

Sample item 3

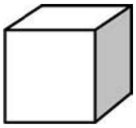
Nick wants to pave the rectangular patio of his new house with bricks. The patio has a length of 5.25 metres and a width of 3.00 metres. One box of bricks can pave 2 square metres.

Calculate the number of boxes of bricks Nick needs for the entire patio.

For this item, the student must perform two steps to arrive at a correct solution. First, the student must find the area of the patio. For the second step, the student must divide the number of square metres by two in order to find the total number of boxes of bricks. This addresses multiple processes in understanding what must be done, devising a strategy and performing the calculations. If a student is only able to successfully find the area, proficiency 1b is addressed. Proficiency 1a is addressed if a student does both steps correctly.

Sample item 4

Susan likes to build blocks using small cubes like the one shown in the following diagram:



Small cube

Susan will build a block as shown in Diagram A below:

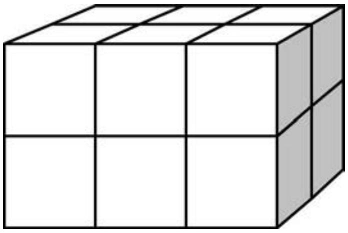


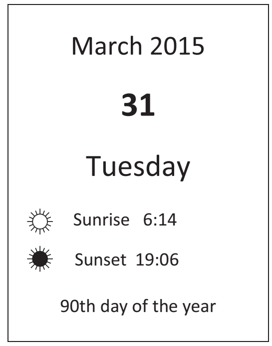
Diagram A

How many small cubes will Susan need to build the block shown in Diagram A?

For this item, the student needs to determine the number of small cubes needed to build the larger block. In doing this, the process "applying mathematical facts, rules, algorithms and structures when finding solutions" is addressed. Since this is a simple, one-step problem, it meets the requirements of proficiency 1c.

Sample item 5

The picture represents one page of a calendar.



How much time will pass on that day from sunrise until sunset?

A. 12 hours and 52 minutes

B. 13 hours and 8 minutes

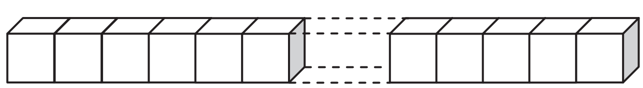
C. 13 hours and 32 minutes

D. 13 hours and 52 minutes

For this item, the student must determine the elapsed time. To solve this successfully, students must recognise the modification to the normal subtraction algorithm when regrouping. Because this recognition is required, even though it is one single operational step, the thought process involved makes this a 1a item rather than 1b or 1c.

Sample item 6

A cube of volume 1 m^3 has been cut off into small cubes of edge length 1 cm. If we put those small cubes one after another, as shown in the picture, we would get a square prism.



Which of the following statements are true? Mark T when the statement is true or F when it is false.

The volume of this square prism is 100 times larger than the volume of the original cubes.	**T**	**F**
One of the edges of this square prism has length 10 km.	**T**	**F**

For this item, the student must demonstrate an understanding of the concept of volume. The first statement requires no calculations at all, only simple reasoning. It is proficiency Level 1a, because there is modelling hidden here. We do not see the large cube in the picture which prevents this item from being 1b. The second statement requires a calculation in order to answer correctly. Students who go by "common sense" usually choose the wrong answer. The student has to ignore his or her intuitive judgment and pick the mathematical way of dealing with the problem by determining the number of small cubes and recognising the change in units. This statement is proficiency Level 2.

Notes

1. In some countries, "mathematical tools" can also refer to established mathematical procedures such as algorithms. For the purposes of the PISA framework, "mathematical tools" refers only to the physical and digital tools described in this section.

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